ON THE NUMBER OF CATENARIES THAT MAY BE DRAWN THROUGH TWO FIXED POINTS.*

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1. We shall see that there are three cases that come under our consideration :

I. Two catenaries may be drawn through the two points ;

II. Only one may be drawn through these points ;

III. No catenary can be drawn through the two points.

It was shown (Vol. X, p. 85) that

$$y = \tfrac{1}{2}m\left(e^{(x-x_0')/m} + e^{-(x-x_0')/m}\right), \tag{1}$$

and

$$\frac{x - x_0'}{m} = \log \text{nat}\left[\frac{y + \sqrt{y^2 - m^2}}{m}\right]. \tag{2}$$

Substitute in (2) first the coordinates of P_0 for x and y, and then those of P_1 ; subtract the latter result from the former, x_0' vanishes, and we have a transcendental equation in m, which we must investigate and find the roots.

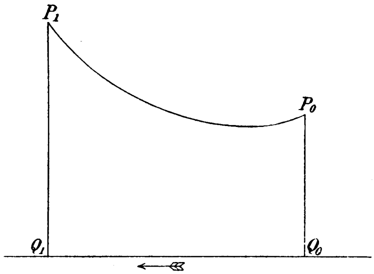

Fig. I.

2. We assume that $y_1 \geqq y_0$. From the equation of the catenary

$$y_0 = \tfrac{1}{2}m\left(e^{(x_0-x_0')/m} + e^{-(x_0-x_0')/m}\right),$$

and

$$y_0^2 - m^2 = \tfrac{1}{4}m^2\left(e^{(x_0-x_0')/m} - e^{-(x_0-x_0')/m}\right)^2.$$

* The present paper is a continuation of the papers that have appeared in the Annals of Mathematics, Vol. IX, p. 179, and Vol. X, p. 81. References to these papers will be made by giving simply the volume and page, or by giving the page alone when the current volume is referred to. The results of this article are in a great measure due to some notes on the Calculus of Variations that were given by Prof. Schwarz at Berlin during the summer semester, 1892. Some of the results were discovered by Goldschmidt, assistant of Gauss ; but as presented here the problem is discussed with greater fullness, and is not only of much importance in itself, but it also serves as an excellent example to illustrate how inexact the former methods of the Calculus of Variations were.

Therefore

$$\sqrt{y_0^2 - m^2} = \pm \tfrac{1}{2}m\,(e^{(x_0-x_0')/m} - e^{-(x_0-x_0')/m})\,; \tag{I}$$

and from this relation it is seen that $\sqrt{y_0^2 - m^2}$ has a positive or a negative sign according as $x_0 - x_0' \gtrless 0$. Hence, also,

$$\frac{x_0 - x_0'}{m} = \pm \log \text{nat} \left[\frac{y_0 + \sqrt{y_0^2 - m^2}}{m}\right]. \tag{a}$$

3. We must first show that such a figure as the one given cannot exist in the present discussion.

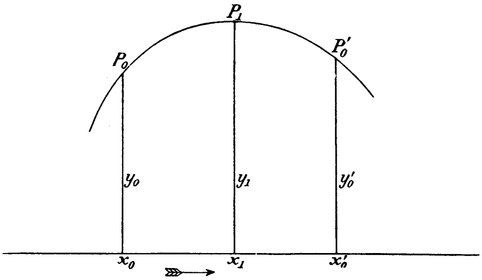

FIG. II.

Without regarding the figure we know that

$$y_1 = \tfrac{1}{2}m\,(e^{(x_1-x_0')/m} + e^{-(x_1-x_0)/m}), \quad x_1 - x_0' > 0\,.$$

That $x_1 - x_0'$ is necessarily positive is seen from the fact that the ordinate $y_0 = m$ corresponds to the value x_0' and is a minimum (p. 86).

Suppose next that $x_0' > x_1$.

By hypothesis $y_1 \geqq y_0$, and further $m \leqq y_0$, and consequently $m \leqq y_1$. The form of the curve is then that given in the figure; and we have within the interval x_0 to x_0' a value of x, for which the ordinate y is greater than it is at the end points.

y must therefore have within this interval a maximum value. But we have shown above (p. 86) that there is no maximum value of y.

Hence

$$\sqrt{y_1^2 - m^2} = +\tfrac{1}{2}m\,(e^{(x_1-x_0')/m} - e^{-(x_1-x_0')/m})\,,$$

and cannot have the *minus* sign as in equation (I). Hence

$$\frac{x_1 - x_0'}{m} = + \log \text{nat} \left[\frac{y_1 + \sqrt{y_1^2 - m^2}}{m}\right]. \tag{b}$$

4. Eliminate x_0' from (a) and (b) and noting that in (a) there is the $\pm$ sign, we have two different functions of m, which may be written:

$$f_1(m) = \log \text{nat} \left[\frac{y_1 + \sqrt{y_1^2 - m^2}}{m}\right] - \log \text{nat} \left[\frac{y_0 + \sqrt{y_0^2 - m^2}}{m}\right] - \frac{x_1 - x_0}{m},$$

and

$$f_2(m) = \log \text{nat} \left[\frac{y_1 + \sqrt{y_1^2 - m^2}}{m}\right] + \log \text{nat} \left[\frac{y_0 + \sqrt{y_0^2 - m^2}}{m}\right] - \frac{x_1 - x_0}{m},$$

two equations of a transcendental nature, which we have now to consider. We must see whether $f_1(m) = 0, f_2(m) = 0$ have roots with regard to m; that is, whether it is possible to give m positive real values, so that the equations $f_1(m) = 0, f_2(m) = 0$ will be satisfied: and if we can so determine m, we must then see whether the values of x_0' which may be then derived from equations (a) and (b) are one-valued.

The first derivative of $f_1(m)$ is

$$f_1'(m) = \frac{1}{m}\left[-\frac{y_1}{\sqrt{y_1^2 - m^2}} + \frac{y_0}{\sqrt{y_0^2 - m^2}} + \frac{x_1 - x_0}{m}\right]. \tag{c}$$

On the right hand side of this expression $1/m$ is positive, also $(x_1 - x_0)/m$ is positive, and

$$\frac{1}{\sqrt{1 - m^2/y_0^2}} - \frac{1}{\sqrt{1 - m^2/y_1^2}} \text{ is positive, if } y_1 > y_0;$$

so that $f_1'(m)$ is positive in the interval $0 \ldots y_0$.

Further

$$f_1(0) = \log \text{nat}\, 2y_1 - \log \text{nat}\,(m = 0) - \log \text{nat}\, 2y_0 + \log \text{nat}\,(m = 0)$$
$$- (x_1 - x_0)/(m = 0) = -\infty.$$

5. It is further seen that $f_1(m)$ continuously increases within the interval $0 \ldots y_0$, so that $-\infty$ is the least value that $f_1(m)$ can take.

Again

$$f_1(y_0) = \log \text{nat} \left[\frac{y_1 + \sqrt{y_1^2 - y_0^2}}{y_0}\right] - \frac{x_1 - x_0}{y_0}. \tag{II}$$

Then if

I. $f_1(y_0) < 0$, $f_1(m)$ has no root;

II. $f_1(y_0) = 0$, $f_1(m)$ has one root, viz., $m = y_0$;

III. $f_1(y_0) > 0$, $f_1(m)$ has a root, $m_1 < y_0$.

When

$$f_1(y_0) < 0\,, \quad P_1 \text{ is outside of the catenary ;}$$

$$f_1(y_0) = 0\,, \quad P_1 \text{ is on the catenary ;}$$

$$f_1(y_0) > 0\,, \quad P_1 \text{ is within the catenary.}$$

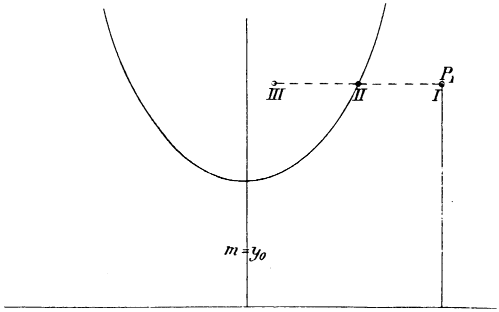

FIG. III.

This may be shown as follows :

$$y = \tfrac{1}{2}y_0\left(e^{(x-x_0)/y_0} + e^{-(x-x_0)/y_0}\right),$$

since when $y = m$, $x = x_0'$; and, therefore, when $y = y_0 = m$, $x = x_0$. We also have

$$y^2 - y_0^2 = \tfrac{1}{4}y_0^2\left(e^{(x-x_0)/y_0} - e^{-(x-x_0)/y_0}\right)^2.$$

Hence

$$\sqrt{y^2 - y_0^2} = \pm\,\tfrac{1}{2}y_0\left(e^{(x-x_0)/y_0} - e^{-(x-x_0)/y_0}\right);$$

the positive sign to be taken when $x > x_0$, and the negative sign, when $x < x_0$.

We also have

$$x - x_0 = y_0 \log \text{nat}\left[\frac{y + \sqrt{y^2 - y_0^2}}{y_0}\right].$$

Comparing this equation with equation (II) above, and noticing the figure, it is seen that, when

$$x_1 - x_0 = y_0 \log \text{nat}\left[\frac{y_1 + \sqrt{y_1^2 - y_0^2}}{y_0}\right], \text{ then } P_1 \text{ is on catenary,}$$

$$x_1 - x_0 > y_0 \log \text{nat}\left[\frac{y_1 + \sqrt{y_1^2 - y_0^2}}{y_0}\right], \text{ then } P_1 \text{ is outside catenary,}$$

$$x_1 - x_0 < y_0 \log \text{nat}\left[\frac{y_1 + \sqrt{y_1^2 - y_0^2}}{y_0}\right], \text{ then } P_1 \text{ is within catenary.}$$

Hence, when $f_1(y_0) > 0$, there is one and only one real root in the interval

$0 \ldots y_0$, and we can draw through the points P_1 and P_0 a catenary, for which the abscissa of the lowest point is $< x_0$.

6. *The Discussion of* $f_2(m) = 0$. We saw (Art. 4) that

$$f_2(m) = \log\left[\frac{y_1 + \sqrt{y_1^2 - m^2}}{m}\right] + \log\left[\frac{y_0 + \sqrt{y_0^2 - m^2}}{m}\right] - \frac{x_1 - x_0}{m}.$$

Therefore

$$f_2'(m) = -\frac{1}{m^2}\left\{\frac{y_1 m}{\sqrt{y_1^2 - m^2}} + \frac{y_0 m}{\sqrt{y_0^2 - m^2}} - (x_1 - x_0)\right\}.$$

When m changes from 0 to y_0, $\sqrt{y_0^2/m^2 - 1}$ continuously decreases, and consequently $y_0/\sqrt{y_0^2/m^2 - 1}$ becomes greater and greater. Hence if the expression $-m^2 f_2'(m)$ takes the value 0, it takes it only once in the interval from 0 to y_0.

That this expression does take the value 0 within this interval is seen from the fact that, for $m = 0$, $-m^2 f_2'(0) = -(x_1 - x_0)$, where $x_1 - x_0 > 0$, so that $-m^2 f_2'(0)$ has a negative value; but, for $m = y_0$, $-m^2 f_2'(0) = +\infty$, so that the expression must take the value zero between these two values of m.

Let μ be this value of m which satisfies the equation, so that

$$\frac{y_1 \mu}{\sqrt{y_1^2 - \mu^2}} + \frac{y_0 \mu}{\sqrt{y_0^2 - \mu^2}} - (x_1 - x_0) = 0, \tag{A}$$

which is an algebraical equation of the 8th degree in μ; or, an algebraical equation of the 4th degree in μ^2.

7. *An approximate geometrical construction for the root that lies between* 0 *and* y_0. In the figure it is seen that the triangles $P_0Q_0A_0$ and $P_0Q_0C_0$ are

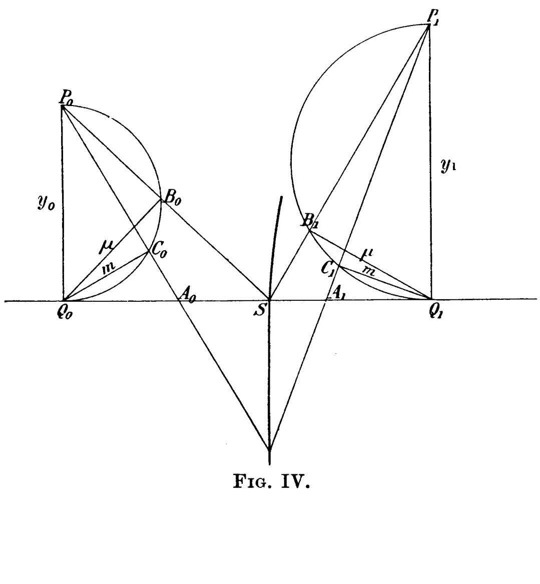

FIG. IV.

similar, as are also the triangles $P_1Q_1A_1$ and $P_1Q_1C_1$. Hence, if m is the length of the line $Q_0C_0 = Q_1C_1$, we have

$$Q_0A_0 = \frac{y_0 m}{\sqrt{y_0^2 - m^2}}, \text{ and } Q_1A_1 = \frac{y_1 m}{\sqrt{y_1^2 - m^2}}.$$

By taking equal lengths $Q_0C_0 = Q_1C_1$ on the two semicircles and prolonging P_0C_0 and P_1C_1 until they intersect, we have as the locus of the intersections a certain curve. This curve must intersect the axis of the x in a point S, say.

Noting that

$$Q_0S + Q_1S = Q_0Q_1 = x_1 - x_0,$$

it follows that

$$\frac{y_0 \cdot Q_0B_0}{\sqrt{y_0^2 - Q_0B_0^2}} + \frac{y_1 \cdot Q_1B_1}{\sqrt{y_1^2 - Q_1B_1^2}} = x_1 - x_0,$$

which compared with the equation (A) above, shows that

$$Q_0B_0 = Q_1B_1 = \mu.$$

8. *The curves represented by the equations* $f_1(m)$ *and* $f_2(m)$.

Equation (c) gives $f_1'(y_0) = \infty$; that is, the tangent to the curve $f_1(m)$ at the point y_0 is parallel to the axis of y.

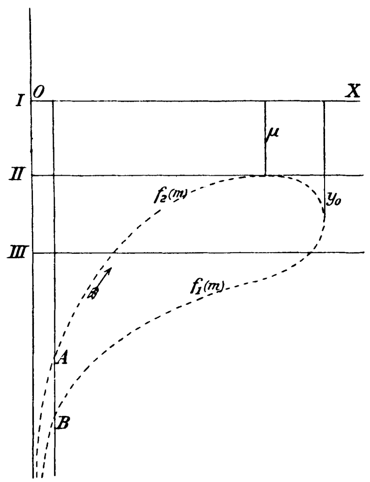

FIG. V.

Further $f_1(0) = -\infty$, so that the negative half of the axis of y is asymptotic to the curve $f_1(m) = 0$. The branch of the curve is here algebraic, since $f_1(m) = 0$ for $m = 0$ is purely algebraically infinite.

9. *Consider next the curve* $f_2(m) = 0$.

It is seen that $f_1(y_0) = f_2(y_0)$; and also $f_2'(y_0) = -\infty$, so that the tangent at this point is also parallel to the axis of the y. Further the negative half of the axis of the y is an asymptote to the curve; but the branch of $f_2(m) = 0$ is transcendent at the point $m = 0$; because logarithms enter in the development of this function in the neighborhood of $m = 0$, as may be seen as follows:

$$f_2(m) = \log\left[\frac{y_1 + \sqrt{y_1^2 - m^2}}{m}\right] + \log\left[\frac{y_0 + \sqrt{y_0^2 - m^2}}{m}\right] - \left[\frac{x_1 - x_0}{m}\right]$$

$$= -\left[\frac{x_1 - x_0}{m}\right] - 2\log m + P(m),$$

where $P(m)$ denotes a power series in positive and integral ascending powers of m. Hence the function behaves in the neighborhood of $m = 0$ as a logarithm.

10. We saw that

$$f_2'(m) = -\frac{1}{m^2}\left[\frac{y_1 m}{\sqrt{y_1^2 - m^2}} + \frac{y_0 m}{\sqrt{y_0^2 - m^2}} - (x_1 - x_0)\right].$$

For the value $m = \mu$, the expression within the brackets is zero, and when $m = 0$, this expression becomes $-(x_1 - x_0)$, and is negative. As seen above in the interval $m = 0$ to $m = y_0$, the expression

$$\left[\frac{y_1 m}{\sqrt{y_1^2 - m^2}} + \frac{y_0 m}{\sqrt{y_0^2 - m^2}} - (x_1 - x_0)\right]$$

becomes greater and greater, so that between the values $m = \mu$ and $m = y_0$, it is positive; and between the values $m = 0$ and $m = \mu$, it is negative.

Furthermore $f_2'(m)$ is positive between $m = 0$ and $m = \mu$, and negative between $m = \mu$ and $m = y_0$.

Hence $f_2(m)$ increases between $m = 0$ and $m = \mu$, and decreases between $m = \mu$ and $m = y_0$; and consequently $f_2(\mu)$ is a maximum.

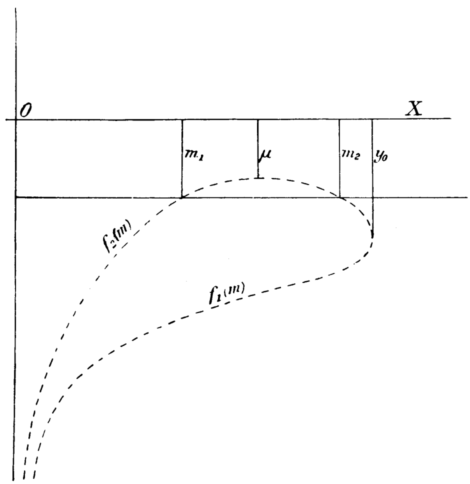

FIG. VI.

11. We must consider the function $f_2(m)$ when m is given different values, and see how many catenaries may be laid between the points P_0 and P_1.

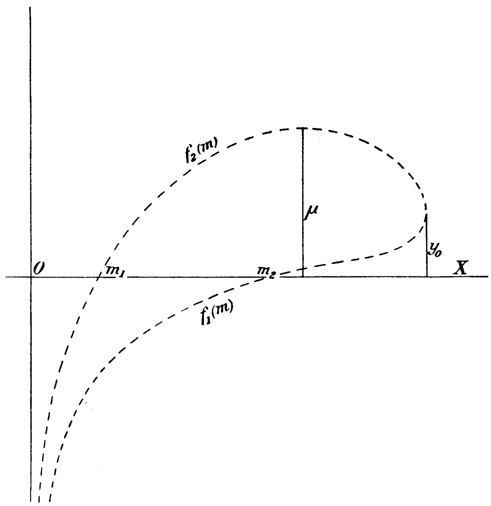

FIG. VII.

We have :

CASE I. $f_2(\mu) < 0$.

In this case $f_2(m)$ is nowhere zero, and there is no root of $f_2(m)$ which we can use.

There is also no root of $f_1(m)$, since

$$f_2(y_0) < 0 \text{ and } f_2(y_0) = f_1(y_0)$$

so that $f_1(y_0) < 0$, and there is no root (see Art. 5).

CASE II. $f_2(\mu) = 0$.

All values of m other than μ cause $f_2(m)$ to be negative, so that there is one and only one root of the equation $f_2(m) = 0$, and consequently only one catenary.

In this case $f_1(m)$ can never be zero; since $f_2(y_0) < 0$, and $f_1(y_0) = f_2(y_0)$, so that $f_1(y_0) < 0$, with the result similar to that in Case I.

CASE III. $f_2(\mu) > 0$.

We have here two catenaries. One root of $f_2(m) = 0$ lies between 0 and μ, and *often* another between μ and y_0, as we may see as follows :—

$$f_2(+0) = -\infty \text{ and } f_2(\mu) > 0 .$$

Since $f_2(m)$ continuously increases in the interval $+0 \ldots \mu$, it can take the value 0 only once within this interval.

In the interval $\mu \ldots y_0$, $f_2(m)$ continuously decreases, so that if $f_2(y_0) > 0$, there is no root of $f_2(m) = 0$ within this interval; but if $f_2(y_0) \overline{\gtrless} 0$, then there

is one and only one root within this interval, and in the latter case there are two catenaries.

We must next consider the roots of $f_1(m)$. When $f_2(y_0) < 0$, then is $f_1(y_0) < 0$, so that there is no root of $f_1(m) = 0$. But when $f_2(y_0) = 0$, then $f_1(y_0) = 0$; and $f_1(m) = 0$ has the root $m = y_0$ which was given above.

Therefore:

A) When $f_2(y_0) < 0, f_2(m)$ has two roots; and when $f_2(y_0) = 0, f_2(m)$ has a root in addition to the root which belongs to $f_2(y_0) = f_1(y_0)$.

B) But when $f_2(y_0) > 0$, then there is only one root for $f_2(m) = 0$, which lies between $0 \ldots \mu$; this root is denoted by m_1.

12. From the formulæ (Art. 4) for $f_1(m)$ and $f_2(m)$, we have

$$f_2(m) = f_1(m) + 2 \log \left[\frac{y_0 + \sqrt{y_0^2 - m^2}}{m}\right].$$

We consider m within the interval $0 \ldots y_0$; for

$$m = 0, \frac{y_0 + \sqrt{y_0^2 - m^2}}{m} = \infty;$$

and for

$$m = y_0, \frac{y_0 + \sqrt{y_0^2 - m^2}}{m} = 1.$$

Consequently within this interval $\log \left[\frac{y_0 + \sqrt{y_0^2 - m^2}}{m}\right]$ is positive, and therefore also $f_2(m) > f_1(m)$; and since $f_2(m_1) = 0$, it follows that $f_1(m_1) < 0$.

On the other hand $f_1(y_0) = f_2(y_0)$; and since $f_2(y_0) > 0$, we have $f_1(y_0) > 0$.

Moreover, within the interval $0 \ldots y_0$, $f_1(m)$ continuously increases, and $f_1(+0) < 0$, so that within the interval $0 \ldots m_1$, $f_1(m)$ has no root, and within the interval $m_1 \ldots y_0$ one root.

Hence under B) $f_2(m)$ has a root m_1 within the interval $0 \ldots \mu$, and only one root, and $f_1(m)$ has a root between m_1 and y_0 and only one; making a total under the heading B) of two catenaries.

We have the following summary:—

1°. $f_2(m) < 0$, no catenary;

2°. $f_2(m) = 0$, one catenary;

3°. $f_2(m) > 0$, two catenaries.

13. *On the consideration of the intersection of the tangents drawn to the catenary at the points P_0 and P_1.*

CASE I. As shown above there is here no catenary, so that the consideration of the tangents is without interest.

CASE II. $f_2(\mu) = 0$.

Here the catenary enjoys the remarkable property that the tangents drawn at the points P_0 and P_1 intersect on the x-axis. In order to show this we must

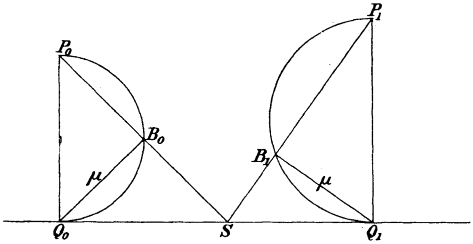

FIG. VIII.

go back to the construction of the tangents at the points P_0 and P_1. It was seen (p. 87) that points B_0 and B_1 were found on the semi-circumferences $P_0B_0Q_0$ and $P_1B_1Q_1$ such that $Q_0B_0 = Q_1B_1$ ($m = \mu$ in this case); and that then the lines P_0B_0 and P_1B_1 were the required tangents, which intersect on the x-axis (see Art. 7).

CASE III. $f_2(\mu) > 0$.

A) $f_2(y_0) \gtreqless 0$.

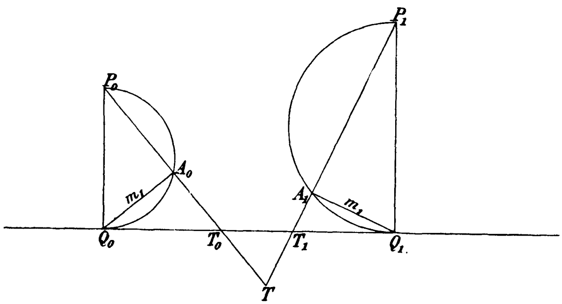

FIG. IX.

Then, as already shown, $f_2(m) = 0$ has two roots, one of which lies between 0 and μ, and the other between μ and y_0.

Let these roots be m_1 and m_2 respectively. For the root m_1, we have

$$Q_0T_0 = \frac{y_0m_1}{\sqrt{y_1^2 - m_1^2}}; \quad Q_1T_1 = \frac{y_1m_1}{\sqrt{y_1^2 - m_1^2}}.$$

We assert that here the intersection of the tangents at P_0 and P_1 lies on the other side of the x-axis.

In order to show this we need only prove that

$$Q_0T_0 + Q_1T_1 < Q_0Q_1.$$

This is shown as follows :—

$$f_2'(m_1) = -\frac{1}{m_1^2}\left\{\frac{y_1 m_1}{\sqrt{y_1^2 - m_1^2}} + \frac{y_0 m_1^2}{\sqrt{y_0^2 - m_1^2}} - (x_1 - x_0)\right\}.$$

Now since $f_2'(m)$ within the interval $0 \ldots \mu$ is positive, and since m_1 lies within this interval, it follows that $f_2'(m_1)$ is positive.

Therefore $-(m_1)^2 f_2'(m_1)$ is negative, and consequently $Q_0T_0 - Q_1T_1 - Q_0Q_1$ is negative.

REMARK. In this consideration the whole interpretation depends upon the fact that the root lies in the interval $0 \ldots \mu$, and the same discussion is applicable to case B) where $f_2(y_0) > 0$, and where the root lies between $0 \ldots \mu$.

15. *On the consideration of the root* m_2.

1°. When $f_2(y_0) \leqq 0$.

The root lies within the interval $\mu \ldots y_0$. $f_2'(m)$ is negative within this interval ; therefore $-m^2 f_2'(m)$ is positive, and consequently

$$\frac{y_1 m^2}{\sqrt{y_1^2 - m_2^2}} + \frac{y_0 m^2}{\sqrt{y_0^2 - m_2^2}} - (x_1 - x_0) > 0 ;$$

$$\therefore \quad Q_0T_0 + Q_1T_1 > Q_0T_0 ;$$

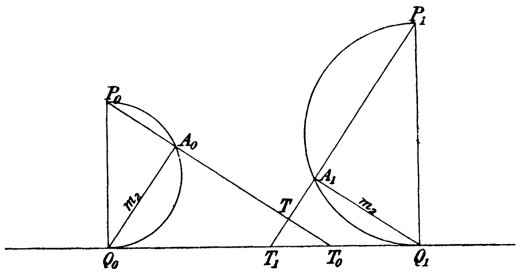

FIG. X.

so that T is on the same side of the x-axis as the curve.

2°. When $f_2(y_0) > 0$; then the root m_2 is a root of the equation $f_1(m) = 0$, so that we have here to consider the sign of

$$\frac{y_1 m_2}{\sqrt{y_1^2 - m_2^2}} + \frac{y_1 m_2}{\sqrt{y_0^2 - m_2^2}} - (x_1 - x_0)$$

within the interval $0 \ldots y_0$.

We have proved that within this interval $f_1'(m)$ is positive, and since

$$f_1'(m_2) = -\frac{1}{m_2^2}\left[\frac{y_1 m_2}{\sqrt{y_1^2 - m_2^2}} - \frac{y_0 m_2}{\sqrt{y_0^2 - m_2^2}} - (x_1 - x_0)\right]$$

is positive, it follows that

$$\left[\frac{y_1 m_2}{\sqrt{y_1^2 - m_2^2}} - \frac{y_0 m_2}{\sqrt{y_0^2 - m_2^2}} - (x_1 - x_0)\right]$$

is negative. Hence

$$\frac{y_1 m_2}{\sqrt{y_1^2 - m_2^2}} - \frac{y_0 m_2}{\sqrt{y_0^2 - m_2^2}} < (x_1 - x_0)\,.$$

And consequently

$$\frac{y_0 m_2}{\sqrt{y_0^2 - m_2^2}} - \frac{y_1 m_2}{\sqrt{y_1^2 - m_2^2}} > (x_1 - x_0)\,.$$

Since $\frac{y_1 m_2}{\sqrt{y_1^2 - m_2^2}}$ is a positive quantity, it follows *a fortiori* that

$$\frac{y_1 m_2}{\sqrt{y_1^2 - m_2^2}} + \frac{y_0 m_2}{\sqrt{y_0^2 - m_2^2}} > (x_1 - x_0)\,,$$

and the intersection lies on the same side of the x-axis as the curve.

16. *Lindelöf's Theorem* (1860).

If we suppose the catenary to revolve around the x-axis, as also the lines $P_0 T$ and $P_1 T$, then the surface generated by the revolution of the catenary is

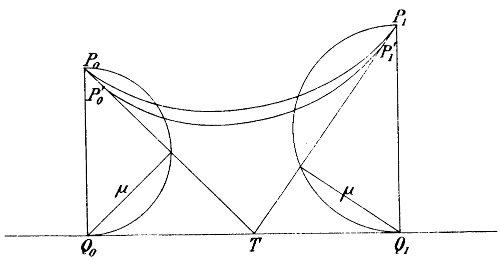

Fig. XI.

equal to the sum of the surfaces generated by the revolution of the two lines $P_0 T$ and $P_1 T$ about the x-axis.

Suppose that with T as *center of similarity* (Aehnlichkeitspunkt), the curve $P_0 P_1$ is subjected to a strain so that P_0 goes into the point P_0', and P_1 into the point P_1', the distance $P_0 P_0'$ being very small and equal, say, to $a = P_1 P_1'$. Then

$$P_0 T : P_0' T = 1 : 1 - a\,.$$

For the sake of abbreviation, let

M_0 denote the surface generated by $P_0 T$, M_0' that generated by $P_0' T$,
M_1 " " " " " $P_1 T$, M_1' " " " $P_1' T$,
S that by the catenary $P_0 P_1$, S' that by the catenary $P_0' P_1'$.

From the nature of the strain the tangents P_0T and P_1T are tangents to the new curve at the points P_0' and P_1', so that we may consider $P_0P_0'P_1'P_1$ as a variation of the curve P_0P_1.

It is seen that

$$S : S' = 1 : (1-\alpha)^2 ;$$

$$M_0 : M_0' = 1 : (1-\alpha)^2 ;$$

$$M_1 : M_1' = 1 : (1-\alpha)^2 .$$

Now from the figure we have as the surface of rotation of $P_0P_0'P_1'P_1$

$$(M_0 - M_0') + S' + (M_1 - M_1') + ((\alpha^2)) = S,$$

where $((\alpha^2))$ denotes a variation of the second order.

Therefore

$$S - S' = (M_0 - M_0') + (M_1 - M_1') + ((\alpha^2)) .$$

Hence

$$S[1-(1-\alpha)^2] = M_0[1-(1-\alpha)^2] + M_1[1-(1-\alpha)^2] + ((\alpha^2)),$$

and consequently

$$2\alpha S = 2\alpha M_0 + 2\alpha M_1 + ((\alpha^2)),$$

or finally

$$S = M_0 + M_1 ,$$

a result which is correct to a differential of the first order.

In a similar manner

$$S' = M_0' + M_1' ;$$

so that

$$S - S' = (M_0 - M_0') + (M_1 - M_1'),$$

or

$$S = (M_0 - M_0') + S' + (M_1 - M_1')$$

is an expression which is absolutely correct.

17. *Geometrical Proof.*

We have seen that

$$\frac{y_0\mu}{\sqrt{y_0^2-\mu^2}} + \frac{y_1\mu}{\sqrt{y_1^2-\mu^2}} - (x_1 - x_0) = 0 , \tag{1}$$

and (see Fig. IV, Art. 7)

$$P_0S = \frac{y_0^2}{\sqrt{y_0^2-\mu^2}} ; \quad P_1S = \frac{y_1^2}{\sqrt{y_1^2-\mu^2}} . \tag{2}$$

The surfaces of the two cones are, therefore, equal to

$$\frac{y_0 \cdot y_0^2\pi}{\sqrt{y_0^2-\mu^2}}, \text{ and } \frac{y_1 \cdot y_1^2\pi}{\sqrt{y_1^2-\mu^2}} .$$

The surface generated by the catenary is

$$\int_{x_0}^{x_1} 2y\pi ds\,.$$

In the catenary $ds = \frac{y}{m}\,.\,dx$ (see p. 87), so that

$$\int_{x_0}^{x_1} 2y\pi ds = \int_{x_0}^{x_1} \frac{2y^2\pi dx}{m} = \pi\,.\,2\int_{x_0}^{x_1} \frac{m^2}{4}[e^{2(x-x_0')/m} + 2 + e^{-2(x-x_0')/m}]\,dx/m$$

$$= \tfrac{1}{4}\pi m^2\,[e^{2(x-x_0')/m} - e^{-2(x-x_0')/m} + 4x/m]_{x_0}^{x_1}$$

$$= \pi\,[\tfrac{1}{2}m\,(e^{(x-x_0')/m} + e^{-(x-x_0')/m})\,.\,\tfrac{1}{2}m\,(e^{(x-x_0')/m} - e^{-(x-x_0')/m}) + mx]_{x_0}^{x_1} \quad \text{(A)}$$

$$= \pi\,[\pm\, y\sqrt{y^2 - m^2} + mx]_{x_0}^{x_1}$$

$$= \pi\,[y_1\sqrt{y_1^2 - m^2} + y_0\sqrt{y_0^2 - m^2} + m\,(x_1 - x_0)]\,, \quad \text{(B)}$$

where we have taken the $+$ sign with $y_0\sqrt{y_0^2 - m^2}$ because $x_0 - x_0'$ is negative, hence $e^{(x-x_0')/m} - e^{-(x-x_0')/m}$ in (A) is negative. But from (1)

$$x_1 - x_0 = \frac{y_1\mu}{\sqrt{y_1^2 - \mu^2}} + \frac{y_0\mu}{\sqrt{y_0^2 - \mu^2}}\,.$$

Substituting in (B), we have, after making $m = \mu$, for the area generated by the revolution of the catenary

$$\pi\left[y_1\sqrt{y_1^2 - \mu^2} + \frac{y_1\mu^2}{\sqrt{y_1^2 - \mu^2}} + y_0\sqrt{y_0^2 - \mu^2} + \frac{y_0\mu^2}{\sqrt{y_0^2 - \mu^2}}\right]$$

$$= \pi\left[\frac{y_1^3}{\sqrt{y_1^2 - \mu^2}} + \frac{y_0^3}{\sqrt{y_0^2 - \mu^2}}\right],$$

which as shown above is the sum of the surface areas of the two cones.

18. Let us consider again for a moment Fig. XII, in which the strain is represented. In order to have a minimum surface of revolution the curve which we rotate must satisfy the differential equation of the problem. If then we had a minimum, this would be brought about by the rotation of the catenary, for the catenary is the curve which satisfies the differential equation; but in our figure this curve can produce no minimal surface of revolution for two reasons: 1° because, drawing tangents (it is shown later that an infinite number may be drawn) which intersect on the x-axis, it is seen that the rotation of $P_0'P_1'$ is the same as that of the two lines $P_0'T$ and $P_1'T$, as shown above; so that there are an infinite number of lines that may be drawn between

P_0 and P_1 which give the same surface of revolution as the catenary between these points; 2° because between P_0 and P_1 lines may be drawn which when caused to revolve about the x-axis would produce a smaller surface area than

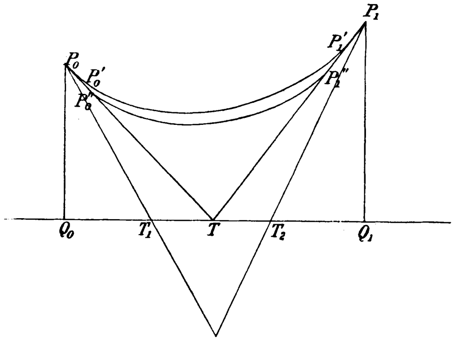

Fig. XII.

that produced by the revolution of the catenary. For the surface generated by the revolution of $P_0'P_1'$ is the same as that generated by $P_0'P_0''P_1''P_1'$. But the straight lines $P_0'P_0''$ and $P_1'P_1''$ do not satisfy the differential equation of the problem, since they are not catenaries. Hence the first variation along these lines is $\gtrless 0$, so that between the points P_0', P_0'' and P_1', P_1'' curves may be drawn whose surface of rotation is smaller than that generated by the straight lines $P_0'P_0''$ and $P_1'P_1''$.

The Case II given above and known as the transition case, i. e., where the point of intersection of the tangents pass from one side to the other side of the axis of the x, affords also no minimal surface, since, as seen above, there are, by varying the variable a, an infinite number of equal surfaces of revolution.

19. In Case III we had two roots of m, which we called m_1 and m_2, where $m_2 > m_1$. We consider first the catenary with parameter m_1. This parameter satisfies the inequality

$$\frac{y_1 m_1}{\sqrt{y_1^2 - m_1^2}} + \frac{y_0 m_1}{\sqrt{y_0^2 - m_1^2}} < x_1 - x_0\,. \tag{A}$$

The equation of the tangent to the curve is

$$\frac{dy}{dx} = \frac{y' - y}{x' - x},$$

where x' and y' are the running coordinates. The intersection of this line with the axis of x is

$$x' - x = -\frac{y}{\frac{dy}{dx}}, \text{ or } x' = x - \frac{y}{\frac{dy}{dx}}; \text{ i. e. } x' = x - m\left[\frac{e^{(x-x_0')/m} + e^{-(x-x_0')/m}}{e^{(x-x_0')/m} - e^{-(x-x_0')/m}}\right].$$

Hence when $x = x_0'$, $x' = -\infty$, and when $x = +\infty$, $x' = +\infty$.

On the other hand dx'/dx is always positive, so that x' always increases when x increases, and the tangent passes from $-\infty$ along the x-axis to $+\infty$, and never passes twice through the same point.

20. It remains yet to show that there are portions of the arc of the catenary, which may be taken between the points P_0 and P_1 in such a way that tangents drawn to the catenary at the end points of these portions of arc intersect on the x-axis.

Let us suppose the point P_0 fixed, while the point P_1' moves from the lower point of the catenary in the direction P_1; the tangent to the catenary at this point moves from $-\infty$ along the x-axis until it reaches the point T_2 (Fig. XII), and then P_1' is at the point P_1, so that the tangent must have passed through the point T_1.

It is thus seen that there are an infinite number of pairs of points on the catenary between the points P_0 and P_1, such that the tangents at any of these pairs of points intersect on the x-axis; and as in Case III, there can be no minimum. Such pairs of points are known as *conjugate points.* Since x' always increases with x, no portion of the x-axis is covered twice by the intersection of the tangents.

Case III, when $m = m_2$; then there is a minimum as Prof. Weierstrass showed in the year 1879. He proved that within a certain region the curves

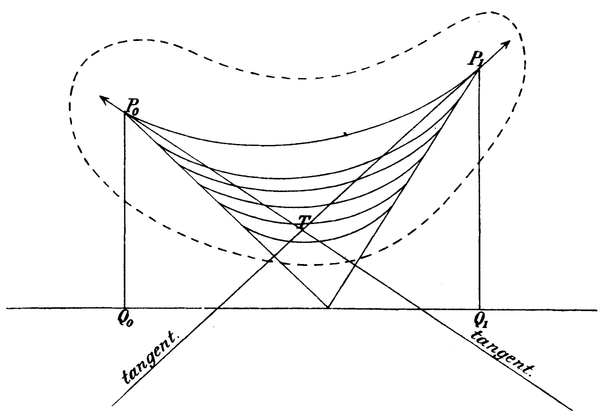

FIG. XIII.

that are caused to exist the one from the other by strains (see Art. 16) having a common point on the x-axis as center of similarity do not cut each other; and from this results the existence of a minimum, as will be seen later.